Bundle gerbes: stable isomorphism and local theory.

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Abstract

We consider the notion of stable isomorphism of bundle gerbes. It has the consequence that the stable isomorphism classes of bundle gerbes over a manifold M are in bijective correspondence with $H^3(M, \mathbb{Z})$. Stable isomorphism sheds light on the local theory of bundle gerbes and enables us to develop a classifying theory for bundle gerbes using results of Gajer on $B\mathbb{C}^\times$ bundles.
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1 Introduction

Recently there has been interest in applying the theory of gerbes to differential geometry [8] and physics [9]. In [11] the first named author described the basic structure of bundle gerbes. In as much as a gerbe is a sheaf of groupoids and a bundle gerbe is a bundle of groupoids, the relationship gerbes and bundle gerbes is analogous to that between locally invertible sheaves and line bundles.

Just as for gerbes, a bundle gerbe over a manifold M has a characteristic class, called its Dixmier-Douady class, which lives in $H^3(M, \mathbb{Z})$ and vanishes precisely when the bundle gerbe is trivial. In [11] the theory of bundle gerbes was shown to be in many ways analogous to that of complex line bundles with the Dixmier-Douady class playing the

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role of the Chern class. One missing link in [11] was the fact that the natural notion of isomorphism of bundle gerbes had the unpleasant property that two bundle gerbes could be non-isomorphic but have the same Dixmier-Douady class. This, of course, is not true for line bundles and their Chern class. In [5] the notion of *stable isomorphism* of bundle gerbes was introduced. This does have the property that two bundle gerbes are stably isomorphic precisely when they have the same Dixmier-Douady class. In the present work we give the precise definition of stable isomorphism and elaborate its properties.

A consequence of the definition of stable isomorphism is a local theory of bundle gerbes analogous to describing a line bundle by transition functions. This description of bundle gerbes is the same as the local theory of gerbes used in [8].

2 Bundle gerbes

Before recalling the definition of bundle gerbe from [11] we need some notation for fibre products and some constructions for $\mathbb{C}^\times$ bundles. Mostly we will be working in the category of smooth manifolds and maps but often these will need to be infinite-dimensional manifolds and in the classifying theory they will be differentiable spaces not manifolds [7]. In the interests of brevity we will just say map and assume the category can be deduced from context.

We will be interested in maps $\pi\colon Y \to M$ which admit local sections. That is, for every $x \in M$ there is an open set U containing x and a local section $s\colon U \to Y$. For want of a better term we will call maps like this locally split. Note that a locally split map is necessarily surjective and that if we are dealing with the smooth category a locally split map is just a submersion. Locally trivial fibrations are, of course, locally split but we will not require $Y \to M$ to be a fibration as we did in [11] because there are a number of important examples in which $Y \to M$ only admits local sections. In particular if $U = \{U_\alpha\}_{\alpha \in I}$ is an open cover over M admitting local sections $s_\alpha\colon U_\alpha \to P$ let Y_U be the disjoint union of all the elements in the open cover, that is the *nerve* of the open cover. Then, of course, $Y_U \to M$ is rarely a fibration.

Let $Y \to M$ be locally split. Then we denote by $Y^{[2]} = Y \times_\pi Y$ the fibre product of Y with itself over π, that is the subset of pairs (y, y') in $Y \times Y$ such that $\pi(y) = \pi(y')$. More generally we denote the pth fold fibre product by $Y^{[p]}$.

Let R and S be principal $\mathbb{C}^\times$ spaces, that is spaces on which $\mathbb{C}^\times$ acts smoothly, freely and transitively. Then there is a notion of a dual space R^*; this is the same space as R but with the inverse $\mathbb{C}^\times$ action. The product $R \times S$ is a principal $\mathbb{C}^\times \times \mathbb{C}^\times$ space. If we quotient $R \times S$ by the subgroup $\{(z, z^{-1}) \mid z \in \mathbb{C}^\times\}$ we obtain the contracted product $R \otimes S$ a principal $\mathbb{C}^\times$ space [2]. These constructions extend in a fibrewise manner to principal $\mathbb{C}^\times$ bundles so that if P and Q are principal $\mathbb{C}^\times$ bundles we can form P^* and $P \otimes Q$. If we replace P and Q by associated complex line bundles in the standard fashion then these two operations are just linear dual and tensor product.

A bundle gerbe over M is a pair (P, Y) where $\pi\colon Y \to M$ is a locally split map and P is a $\mathbb{C}^\times$ bundle $P \to Y^{[2]}$ with a product, that is, a $\mathbb{C}^\times$ equivariant map

$$P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \to P_{(y_1,y_3)}$$

for every (y_1, y_2) and (y_2, y_3) in $Y^{[2]}$. We require the product to be smooth in y_1, y_2 and y_3 but in the interests of brevity we will not state the various definitions needed to make this requirement precise, they can be found in [11]. The product is required to be associative

whenever triple products are defined. We shall often refer to a bundle gerbe (P, Y) as just P. It is shown in [11] that the product defines uniquely an inverse

$$P_{(y_1,y_2)} \to P_{(y_2,y_1)}$$

denoted by $p \mapsto p^{-1}$ and an identity element $1 \in P_{(y,y)}$ which behave as one would expect. For example $(pq)^{-1}1 = q^{-1}p^{-1}$.

Various operations are possible on bundle gerbes. Let (P, Y) be a bundle gerbe over M. Let $\pi\colon Z \to N$ be another locally split map and let $\hat{\phi}\colon Z \to Y$ be a fibre map covering a map $\phi\colon N \to M$. Then there is an induced map $\hat{\phi}^{[2]}\colon Z^{[2]} \to Y^{[2]}$ which can be used to pull-back the bundle $P \to Y^{[2]}$ to a bundle $(\hat{\phi}^{[2]})^{-1}(P) \to Z^{[2]}$. This has an induced product on it and defines a bundle gerbe which we denote, for simplicity, by $(\phi^{-1}(P), Z)$ or $\phi^{-1}(P)$. Two special cases of this are important. The first is when we just have just a map $f\colon N \to M$ and use this to pull-back $Y \to M$ to $f^{-1}(Y) \to N$. The second is when we have $M = N$ and ϕ the identity.

If (P, Y) is a bundle gerbe we can define a new bundle gerbe, (P^*, Y), the dual of (P, Y), by taking the dual of P. Also if (P, Y) and (Q, Z) are two bundle gerbes we can define their product $(P \otimes Q, Y \times_\pi Z)$ where $Y \times_\pi Z$ is the fibre product.

Two bundle gerbes (P, Y) and (Q, Z) are called *isomorphic* if there is an isomorphism $f\colon Y \to Z$, commuting with the projection to M and a bundle isomorphism $g\colon P \to Q$ covering the induced map $f^{[2]}\colon Y^{[2]} \to Z^{[2]}$ and commuting with the bundle gerbe products on Q and P respectively.

If Q is a $\mathbb{C}^\times$ bundle over Y then we can define a bundle gerbe $\delta(Q)$ by $\delta(Q) = \pi_1^{-1}(Q)^* \otimes \pi_2^{-1}(Q)$, that is $\delta(Q)_{(y_1,y_2)} = Q^*_{y_1} \otimes Q_{y_2}$. The bundle gerbe product is induced by the natural pairing

$$Q^*_{y_1} \otimes Q_{y_2} \otimes Q^*_{y_2} \otimes Q_{y_3} \to Q_{y_1^*} \otimes Q_{y_3}.$$

A bundle gerbe which is isomorphic to a bundle gerbe of the form $\delta(Q)$ is called *trivial*. A choice of Q and a bundle gerbe isomorphism $\delta(Q) \simeq P$ is called a *trivialisation*. If Q and R are trivialisations of P then we have natural isomorphisms

$$Q^*_{y_1} \otimes Q_{y_2} \simeq R^*_{y_1} \otimes R_{y_2}$$

and hence

$$Q^*_{y_1} \otimes R_{y_1} \simeq Q^*_{y_2} \otimes R_{y_2}$$

so that the bundle $Q \otimes R$ is the pull-back of a bundle on M. Moreover if Q is a trivialisation and S is a bundle on M then $Q \otimes \pi^{-1}(S)$ is also a trivialisation.

There is a characteristic class $d(P) = d(P, Y) \in H^3(M, \mathbb{Z})$, the Dixmier-Douady class of (P, Y). We recall the definition from [11] as we will need it later. Choose an open cover $\{U_\alpha\}_{\alpha \in I}$ of M for which all intersections are contractible. Let $s_\alpha\colon U_\alpha \to Y$ be sections. These induce sections $(s_\alpha, s_\beta)\colon U_\alpha \cap U_\beta \to Y^{[2]}$. Choose sections $\sigma_{\alpha\beta}$ of $(s_\alpha, s_\beta)^{-1}(P)$ over $U_\alpha \cap U_\alpha$. The bundle gerbe multiplication allows us to compare $\sigma_{\alpha\beta}\sigma_{\beta\gamma}$ and $\sigma_{\alpha\gamma}$ and define a map $g_{\alpha\beta\gamma}\colon U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{C}^\times$ by

$$\sigma_{\alpha\beta}\sigma_{\beta\gamma} = \sigma_{\alpha\gamma} g_{\alpha\beta\gamma}.$$

This map $g_{\alpha\beta\gamma}$ is a $\mathbb{C}^\times$ valued cocycle and defines the Dixmier-Douady class in $H^2(M, \mathbb{C}^\times) \simeq H^3(M, \mathbb{Z})$.

It is shown in [11] that

Theorem 2.1 ([11]). *A bundle gerbe (P, Y) has zero Dixmier-Douady class precisely when it is trivial.*

The construction of the Dixmier-Douady class is natural in the sense that if $Z \to N$ is another locally split map and $\hat{\phi}\colon Z \to Y$ is a fibre map covering $\phi\colon N \to M$ then it is straightforward to check from the definition that

$$d(\phi^{-1}(P), Z) = \phi^*(d(P, Y)). \tag{1}$$

In particular if $M = N$ and ϕ is the identity then

$$d(\phi^{-1}(P)) = d(P). \tag{2}$$

From [11] we also have

Theorem 2.2 ([11]). *If P and Q are bundle gerbes over M then*

1. *$d(P^*) = -d(P)$ and*
2. *$d(P \otimes Q) = d(P) + d(Q)$.*

3 Stable isomorphism of bundle gerbes

Equation (2) shows that there are many bundle gerbes which have the same Dixmier-Douady class but which are not isomorphic. We consider in this section the notion of *stable isomorphism* of bundle gerbes — introduced in [5] — that will have the consequence that two bundle gerbes have the same Dixmier-Douady class if and only if they are stably isomorphic. We will show in a later section that there is a *universal bundle gerbe* with the property that every bundle gerbe over M is stably isomorphic to the pull-back of the universal bundle gerbe under a classifying map.

We begin with

Definition 3.1. Two bundle gerbes (P, Y) and (Q, Z) are called *stably isomorphic* if there are trivial bundle gerbes T_1 and T_2 such that

$$P \otimes T_1 = Q \otimes T_2.$$

There are equivalent definitions of stable isomorphism provided by the following proposition.

Proposition 3.2. *For bundle gerbes (P, Y) and (Q, Z) the following are equivalent.*

1. *P and Q are stably isomorphic*
2. *$P \otimes Q^*$ is trivial*
3. *$d(P) = d(Q)$.*

Proof. Clearly stably isomorphic bundle gerbes have the same Dixmier-Douady class because trivial bundles have the zero Dixmier-Douady class and the Dixmier-Douady class is additive over tensor products. So (1) implies (3). If $d(P) = d(Q)$ then $d(P \otimes Q^*) = d(P) - d(Q) = 0$. Hence $P \otimes Q^*$ is trivial ([11]). So (3) implies (2). Finally if $P \otimes Q^*$ is trivial then $Q \otimes Q^*$ is also trivial as it has zero Dixmier-Douady class and then $P \otimes (Q^* \otimes Q) = Q \otimes (P \otimes Q^*)$ so P and Q are stably isomorphic. So (2) implies (1). $\square$

Note that stable isomorphism for bundle gerbes is analogous to stable isomorphism for vector bundles [1].

From part (3) of Proposition 3.2 we see that stable isomorphism is an equivalence relation. It was shown in [11] that every class in $H^3(M, \mathbb{Z})$ is a the Dixmier-Douady class of some bundle gerbe. Hence we can deduce from Proposition 3.2 that

Theorem 3.3. *The Dixmier-Douady class defines a bijection between stable isomorphism classes of bundle gerbes and $H^3(M, \mathbb{Z})$.*

The following is an important example of stably isomorphic bundle gerbes.

Proposition 3.4. *Let $\pi\colon Z \to M$ and $\pi\colon Y \to M$ be locally split maps and $\phi\colon Z \to Y$ a fibre map covering the identity on M. Let (P, Y) be a bundle gerbe then $(\phi^{-1}(P), Z)$ and (P, Y) are stably isomorphic.*

Proof. This is clear because they have the same Dixmier-Douady class but it is instructive to also show that $\phi^{-1}(P)^* \otimes P$ is trivial. To see this note that the fibre of $(\phi^{-1}(P))^* \otimes P$ at a point $((z_1, y_1), (z_2, y_2))$ in $(Z \times_\pi Y)^{[2]}$ is

$$\phi^{-1}(P)^*_{(z_1,z_2)} \otimes P_{(y_1,y_2)} = P^*_{(\phi(z_1),\phi(z_2))} \otimes P_{(y_1,y_2)}.$$

Whereas if we define Q over $Z \times_\pi Y$ by $Q_{(z,y)} = P_{(\phi(z),y)}$ then $\delta(Q)$ at $((z_1, y_1), (z_2, y_2))$ is

$$P^*_{(\phi(z_1),y_1)} \otimes P_{(\phi(z_2),y_2)}.$$

The gerbe multiplication can be used to define an isomorphism

$$\begin{aligned} P^*_{(\phi(z_1),y_1)} \otimes P_{(\phi(z_2),y_2)} &= P^*_{(\phi(z_1),\phi(z_2))} \otimes P^*_{(\phi(z_2),y_1)} \otimes P_{(\phi(z_2),y_2)} \\ &= P^*_{(\phi(z_1),\phi(z_2))} \otimes P_{(y_1,\phi(z_2))} \otimes P_{(\phi(z_2),y_2)} \\ &= P^*_{(\phi(z_1),\phi(z_2))} \otimes P_{(y_1,y_2)} \end{aligned}$$

□

We remark that an alternative approach to stable isomorphism is to define a stable isomorphism between bundle gerbes P and Q to be a trivialisation of $P \otimes Q^*$. Two bundle gerbes are then stably isomorphic if a stable isomorphism exists. With a little work one can show that stable isomorphisms are composable and that the set of all bundle gerbes over M with stable isomorphisms as morphisms is a category, in fact a two-category [15]. We shall not pursue this approach here.

3.1 Local bundle gerbes

Proposition 3.4 gives us a tool for understanding what a bundle gerbe looks like locally. Consider a bundle gerbe (P, Y) over M and assume that $U = \{U_\alpha\}_{\alpha \in I}$ is an open cover over M with local sections $s_\alpha\colon U_\alpha \to Y$. Let Y_U be the disjoint union of all the elements in the open cover, that is the *nerve* of the open cover. Then the local sections s_α define a map $s\colon Y_U \to Y$ by $s(\alpha, x) = s_\alpha(x)$. By 3.4 the pullback of the bundle gerbe over Y defines a bundle gerbe over Y_U which is stably isomorphic to the original P. If we specialise the definition of bundle gerbes to nerves of open covers we see that it is equivalent to the following:

- a $\mathbb{C}^\times$ bundle $P_{\alpha\beta}$ over each intersection $U_\alpha \cap U_\beta$
- a trivialisation $\theta_{\alpha\beta\gamma}$ of the contracted product $P_{\beta\gamma} \otimes P^*_{\alpha\gamma} \otimes P_{\alpha\beta}$ over $U_\alpha \cap U_\beta \cap U_\gamma$
- the trivialisation satisfies $\delta(\theta) = 1$ over $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$.

Note in the last of these that $\delta(\theta)$ is a section of a $\mathbb{C}^\times$ bundle which is canonically trivial so the statement makes sense. This data can be easily shown to be equivalent to that in [8]. The discussion we have just given shows that every bundle gerbe is stably isomorphic to a 'local' bundle gerbe of this form.

3.2 Other related objects

The geometric structure underlying a bundle gerbe arises in a number of other places which we briefly mention here.

Firstly we remark that it is well-known that the collection of spaces $Y^{[p]}$ form a *simplicial space* [6]. In [3] Brylinski and Maclaughlin define the notion of a *simplicial line bundle*. For the simplicial space $Y^{[p]}$ this is precisely the notion of a bundle gerbe.

Secondly we can regard Y as the set of objects for a non-transitive $\mathbb{C}^\times$ groupoid in which two objects y_1 and y_2 either have no morphism between them if y_1 and y_2 are not in the same fibre or the single morphism (y_1, y_2) if y_1 and y_2 are in the same fibre. Composition of morphisms is $(y_1, y_2) \circ (y_2, y_3) = (y_1, y_3)$. The bundle gerbe P is then a central extension of the groupoid Y called a *twist* [10].

Thirdly in a similar manner to the above any fibre $Y_m = \pi^{-1}(m)$ can be thought of as a groupoid with the morphisms between (y_1, y_2) being just $P_{(y_1,y_2)}$. The bundle gerbe can then be thought of as a bundle of groupoids.

4 Bundle gerbes with connection

In [11] the first author defined the notion of a bundle gerbe connection, curving and curvature which we breifly recall.

Recall from [11] that if $\Omega^p(X)$ denotes the space of p forms on the manifold X, then we have an *exact* complex

$$\Omega^p(M) \xrightarrow{\pi^*} \Omega^p(Y) \xrightarrow{\delta} \Omega^p(Y^{[2]}) \xrightarrow{\delta} \dots . \tag{3}$$

Here $\delta\colon \Omega^p(Y^{[q]}) \to \Omega^p(Y^{[q+1]})$ is the alternating sum of pull-backs $\sum_{j=1}^{q+1}(-1)^j \pi_i^*$ of projections where π_i is the projection map which omits the ith point in the fibre product. In [11] exactness of the complex (3) was proved for the case of $Y \to M$ being a fibration but that proof can be extended to the case that $Y \to M$ is locally split.

Because $P \to Y^{[2]}$ is a $\mathbb{C}^\times$ bundle it has connections. It is shown in [11] that it admits *bundle gerbe connections* that is connections commuting with the bundle gerbe product. A bundle gerbe connection ∇ has curvature F_∇ satisfying $\delta(F_\nabla) = 0$ and hence from the exactness of equation (3) we have that there exists a two-form f on Y, satisfying the 'descent equation'

$$F_\nabla = \pi_1^*(f) - \pi_2^*(f).$$

Such an f is called a *curving* for the connection ∇. The choice of a curving is not unique, from (3) we see that the ambiguity in the choice is precisely the addition of the pull-back of

a two-form from M. Given a choice of curving we then have that $\delta(df) = d\delta(f) = dF_\nabla = 0$ so that we can find some ω, a three-form on M, such that $df = \pi^*(\omega)$. Moreover ω is closed as $\pi^*(d\omega) = ddf = 0$. In [11] it is shown that $\omega/2\pi i$ is a de Rham representative for the Dixmier-Douady class. We call ω the three curvature of the connection and curving.

To define the notion of stable equivalence of bundle gerbes with connection and curving we start by letting $P \to Y$ be a line bundle with connection ∇ and curvature F. The trivial gerbe $\delta(P) \to Y^{[2]}$ has a natural bundle gerbe connection $\delta(\nabla) = \pi_1^*(\nabla) - \pi_2^*(\nabla)$ and curving F. If T is a trivial bundle gerbe we say it has trivial connection and curving if they arise in this way. To extend the definition of stable isomorphism (definition 3.1) to cover the case of bundle gerbes with connection and curving we assume that the trivial bundle gerbes T_1 and T_2, in the definition, have trivial connections and curvings and that the isomorphism in Definition 3.1 preserves connections and curving.

4.1 Bundle gerbes and Deligne cohomology

In [11] it was shown that a bundle gerbe with connection and curving defined an element in the Deligne cohomology group $H^2(M, \underline{\mathbb{C}^\times} \stackrel{d\log}{\to} \Omega^1 \stackrel{d}{\to} \Omega^2)$. In this section we use the notion of stable isomorphism to show

Theorem 4.1. *The stable isomorphism classes of bundle gerbes with connection and curving are equal to* $H^2(M, \underline{\mathbb{C}^\times} \stackrel{d\log}{\to} \Omega^1 \stackrel{d}{\to} \Omega^2)$.

Let us first define the Deligne cohomology class $D(P, \nabla, f)$ of a bundle gerbe P with connection ∇ and curving f. If $\mathcal{U} = \{U_\alpha\}$ is a Leray cover of M then the Deligne cohomology $H^2(M, \underline{\mathbb{C}^\times} \stackrel{d\log}{\to} \Omega^1 \stackrel{d}{\to} \Omega^2)$ can be calculated as the cohomology of the total complex of the double complex:

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow \\
C^2(\mathcal{U}, \underline{\mathbb{C}^\times}) & \stackrel{d\log}{\to} & C^2(\mathcal{U}, \Omega^1) & \stackrel{d}{\to} & C^2(\mathcal{U}, \Omega^2) \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow \\
C^1(\mathcal{U}, \underline{\mathbb{C}^\times}) & \stackrel{d\log}{\to} & C^1(\mathcal{U}, \Omega^1) & \stackrel{d}{\to} & C^1(\mathcal{U}, \Omega^2) \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow \\
C^0(\mathcal{U}, \underline{\mathbb{C}^\times}) & \stackrel{d\log}{\to} & C^0(\mathcal{U}, \Omega^1) & \stackrel{d}{\to} & C^0(\mathcal{U}, \Omega^2)
\end{array}
\tag{4}
$$

Consider an open cover $\mathcal{U} = \{U_\alpha\}$ of M which has all intersections contractible and such that $Y \to P$ has sections s_α over each U_α. Then (s_α, s_β) defines a section of $Y^{[2]}$ over $U_\alpha \cap U_\beta$ which can be used to pull back P to a bundle $P_{\alpha\beta}$. Let $\sigma_{\alpha\beta}$ be a section of this bundle. We have seen that the Dixmier-Douady class is the class $g_{\alpha\beta\gamma}$ defined by $\sigma_{\alpha\beta}\sigma_{\beta\gamma} = g_{\alpha\beta\gamma}\sigma_{\alpha\gamma}$. The connection ∇ can be pulled back by $\sigma_{\alpha\beta}$ to define $A_{\alpha\beta}$ a 1-form on $U_\alpha \cap U_\beta$ and the curving can be pulled-back by the s_α to define two-forms f_α on U_α. The triple $(g_{\alpha\beta\gamma}, A_{\alpha\beta}, f_\alpha)$ is a Deligne cohomology class which we denote by $D(P, \nabla, f)$. It is straightforward to check that it is independent of choices. When it is clear what the connection and curving are we shall often denote the Deligne class by just $D(P)$.

To prove Theorem 4.1 we mimic the proof of Theorem 3.2. Let $D(P)$ denote the Deligne class of a bundle gerbe with connection and curving. Note first that it follows easily from the constructions in [11] that if P and Q are bundle gerbes with connection and curving and we give $P \otimes Q$ and P^* the induced connections and curving then $D(P \otimes Q) = D(P) + D(Q)$ and $D(P^*) = -D(P)$. Assume that we have:

Proposition 4.2. *Let P be a bundle gerbe with connection and curving and zero Deligne class. Then P is a trivial bundle gerbe with trivial connection and curving.*

Consider now P and Q two bundle gerbes with connection and curving and equal Deligne classes then we can form $P \otimes Q^*$. We have $D(P \otimes Q^*) = 0$ so by Proposition 4.2 this is a trivial bundle gerbe with trivial connection and curving. Similarly for $Q \otimes Q^*$. But then we have $P \otimes (Q^* \otimes Q) = Q \otimes (P \otimes Q^*)$ so that P and Q are stably isomorphic.

It remains to prove Proposition 4.2. Assume that (P, Y) is a bundle gerbe over M with $D(P) = 0$. This implies that the Dixmier-Douady class of P is also zero and we may assume that $P = \delta(Q)$ for $Q \to Y$ a $\mathbb{C}^\times$ bundle. Let ∇_P be the connection on P and ∇_Q a connection on Q. Then $\delta(\nabla_Q)$ is another connection on P and hence $\nabla_P - \delta(\nabla_Q)$ is a 1-form with $\delta(\nabla_P - \delta(\nabla_Q)) = 0$. By the exactness of 3 there is a 1-form a on Y such that $\delta(a) = \nabla_P - \delta(\nabla_Q)$ or $\nabla_P = \delta(\nabla_Q - a)$. Replacing ∇_Q by $\nabla_Q - a$ we may assume, without loss of generality that $\nabla_P = \delta(\nabla_Q)$. Consider now the curving f and the curvature F_Q of ∇_Q. We have $\delta(f - F_Q) = 0$ and hence $f - F_Q = \pi^*\phi$ for ϕ a 2-form on M.

Consider the construction of the Deligne class $D(P, \nabla_P, f)$. If we choose sections σ_α of $s_\alpha^{-1}(Q)$ and let $\sigma_{\alpha\beta} = \sigma_\alpha/\sigma_\beta$ we have $g_{\alpha\beta\gamma} = 1$. If a_α is the connection 1-form for $s_\alpha^{-1}(\nabla_Q)$ then $A_{\alpha\beta} = a_\alpha - a_\beta$. The vanishing of the Deligne class implies the existence of $h_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^\times$ and $k_\alpha \in \Omega^1(U_\alpha)$ such that

$$\begin{aligned} g_{\alpha\beta\gamma} &= h_{\beta\gamma} h_{\alpha\gamma}^{-1} h_{\alpha\beta} \\ A_{\alpha\beta} &= -k_\beta + k_\alpha + h_{\alpha\beta}^{-1} dh_{\alpha\beta} \\ f_\alpha &= dk_\alpha \end{aligned}$$

and hence in our case

$$\begin{aligned} 1 &= h_{\beta\gamma} h_{\alpha\gamma}^{-1} h_{\alpha\beta} \\ a_\alpha - a_\beta &= -k_\beta + k_\alpha + h_{\alpha\beta}^{-1} dh_{\alpha\beta} \\ f_\alpha &= dk_\alpha. \end{aligned}$$

We can use the $h_{\alpha\beta}$ to construct a bundle $R \to M$ with a connection ∇_R defined locally by $k_\alpha - a_\alpha$. Pull this back to Y and tensor it by Q. Then $\delta(Q \otimes \pi^{-1}(R)) = \delta(Q) = P$. Moreover the connection $k_\alpha - a_\alpha$ on R has curvature $s_\alpha^*(f_\alpha - F_Q) = s_\alpha^*\pi^*(\phi) = \phi$. Hence the curvature of $\pi^*(\nabla_R)$ is $\pi^*(\phi) = f - F_Q$ and when we tensor ∇_R with ∇_Q the resulting tensor product connection will have curvature f. We have now shown that a bundle gerbe with trivial Deligne cohomology class is a trivial bundle gerbe with trivial connection and curving.

Note 4.1. Note that in the proof of Proposition 4.2 we have shown that if $P = \delta(Q)$ is a trivial bundle gerbe with connection ∇ then there is a connection on Q such that $\delta(Q) = \nabla$. This implies that stable isomorphism classes of bundle gerbes with connection (but not curving) are in bijective correspondence with stable isomorphism classes of bundle gerbes. As the former are determined by their Deligne class in $H^2(M, \underline{\mathbb{C}^\times} \xrightarrow{d\log} \Omega^1)$ and the latter by their Dixmier-Douady class in $H^2(M, \mathbb{C}^\times)$ this is equivalent to the easily proven result that $H^2(M, \underline{\mathbb{C}^\times} \xrightarrow{d\log} \Omega^1) = H^2(M, \mathbb{C}^\times)$.

5 Bundle gerbes and gerbes

In [11] the first author proposed a construction of a presheaf of categories from a bundle gerbe whose sheafification, it was claimed, was a gerbe associated to the bundle gerbe and with the same Dixmier-Douady class. We give here a simpler construction of a gerbe associated to a bundle gerbe which includes the previous case as a subsheaf. Let (P, Y) be a gerbe over M. For any U open in M let $\mathcal{G}(U)$ be a category defined as follows. The objects of $\mathcal{G}(U)$ are the set of all trivialisations of the restriction of (P, Y) to U. That is all pairs (Q, f) where Q is a $\mathbb{C}^\times$ bundle over $Y_U = \pi^{-1}(U) \subset Y$ and $f \colon \delta(P) \to Q_{|Y_U^{[2]}}$ is an isomorphism of bundle gerbes. The morphisms between two objects (Q, f) and (R, g) are all isomorphisms of bundle gerbes which commute with f and g. That is isomorphisms $\phi \colon Q \to R$ of $\mathbb{C}^\times$ bundles, acting as the identity on Y_U such that the induced map

$$\delta(\phi) \colon \delta(Q) \to \delta(R)$$

commutes with the isomorphisms f and g.

Notice that $\mathcal{G}(U)$ is either empty or a groupoid and from section 2 we see that $\mathcal{G}(U)$ is non-empty if Y admits a section over U.

We want to show that $\mathcal{G}$ defines a gerbe. The definition is in [2] except that we will not go for the full generality of assigning to any local homeomorphism a groupoid although that would not be difficult.

First we need to show that it is a sheaf of groupoids. We have already shown that for every open set U we have a (possibly trivial) groupoid. The restriction functor is just exactly the restriction of a trivialisation over Y_U to Y_V if $V \subset U$. For a general sheaf of categories it is possible that the composition of two restriction functors differs from the restriction functor of the composition of two restrictions by a natural transformation. In the case of $\mathcal{G}$ they are clearly equal. This makes $\mathcal{G}$ a presheaf of groupoids ([2] Definition 5.2.1).

To show that $\mathcal{G}$ is a sheaf of groupoids we need to check two patching conditions on objects and morphisms as detailed on pages 191–2 of [2]. Assume we have an open cover $\{U_\alpha\}$ of an open set U. First consider two trivialisations (Q_i, f_i), $i = 1, 2$ in $\mathcal{G}(U)$ with morphisms $\phi_\alpha \colon Q_{1|U_\alpha} \to Q_{2|U_\alpha}$ for each α agreeing on overlaps. Then these clearly patch together to yield a global morphism ϕ and as the ϕ_α commute with the f_i so also does ϕ. Second assume we have trivialisations (Q_α, f_α) in each $\mathcal{G}(U_\alpha)$ and morphisms

$$\phi_{\alpha\beta} \colon Q_{\alpha|U_{\alpha\beta}} \to Q_{\beta|U_{\alpha\beta}}$$

satisfying $\phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1$. Then we need to find a global trivialisation $(Q, f) \in \mathcal{G}(U)$ whose restriction to each U_α is (Q_α, f_α). But this is possible because the condition $\phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1$ allows us to apply the 'clutching construction' to form the bundle Q. It is straightforward to define the trivialisation. Hence we have that $\mathcal{G}$ is a sheaf of categories.

Finally we consider the conditions in [2] Definition 5.2.4 that we need to satisfy to show that $\mathcal{G}$ is a gerbe. First condition (G1). Assume that $\mathcal{G}(U)$ is non-empty. Let (Q, f) be an object in $\mathcal{G}(U)$ and consider the automorphisms of (Q, f). If we think of Q first as a $\mathbb{C}^\times$ bundle on Y_U then the group of all automorphisms is the group of all maps from Y_U to $\mathbb{C}^\times$. However if we require that they also commute with f it is easy to see that they have to be maps that are constant on the fibres of $Y \to M$. Hence they are the group of all maps from U into $\mathbb{C}^\times$ as required to satisfy (G1). Second condition (G2). Let (Q, f)

and (R, g) be objects in $\mathcal{G}(U)$ and let $z \in U$. We have that $Q \otimes R^* = \pi^{-1}(T)$ for some bundle T over U. Choosing a contractible neighbourhood V of z we can trivialise T and this induces an isomorphism from $Q_{|V}$ to $R_{|V}$ as required. Finally the third condition that we can cover M by open sets U such that $\mathcal{G}(U)$ is non-empty follows from the fact that we can cover by open sets over which Y has sections and hence we can trivialise the bundle gerbe locally.

To see that $\mathcal{G}$ has the same Dixmier-Douady class as (P, Y) recall that inside any $\mathcal{G}(U)$ is a subcategory whose objects are induced by choosing sections $s\colon U \to Y$. If we construct the Dixmier-Douady class of the gerbe $\mathcal{G}$ by choosing objects and morphisms of this kind we are constructing the Dixmier-Douady class of the bundle gerbe (P, Y) [11].

Hence we have proven.

Theorem 5.1. *The sheaf of groupoids $\mathcal{G}$ is a gerbe with Dixmier-Douady class equal to that of P.*

Because both stable isomorphism classes of bundle gerbes and equivalence classes of gerbes are classified by $H^3(M, \mathbb{Z})$ it follows that we have established a bijection between stable isomorphism classes of bundle gerbes and equivalence classes of gerbes.

6 Bundle gerbes and $B\mathbb{C}^\times$ bundles.

In this section we recall Gajer's construction of $B\mathbb{C}^\times$ bundles [7] and show that there is a bijection between $B\mathbb{C}^\times$ bundles over M and stable isomorphism classes of bundle gerbes over M. Because both of these sets are in bijective correspondence with $H^3(M, \mathbb{Z})$ the only thing we really need to do is to associate to any $B\mathbb{C}^\times$ bundle a bundle gerbe whose Dixmier-Douady class is the characteristic class of the $B\mathbb{C}^\times$ bundle. This will also give us a classifying theory for bundle gerbes. To this end we need to recall the notion of the *lifting bundle gerbe.*

6.1 Lifting bundle gerbes

Let

$$0 \to \mathbb{C}^\times \to \hat{G} \to G \to 0$$

be a central extension of groups. If $P(M, G)$ is a principal bundle there is a map

$$s\colon P^{[2]} \to G$$

defined by $ps(p, q) = q$ for every p and q in the same fibre of $P \to M$. Denote by $\hat{P}$ the pullback of $\hat{G}$ by s. The group multiplication in $\hat{G}$ commutes with $\mathbb{C}^\times \subset G$ as it is central and makes $(\hat{P}, P)$ a bundle gerbe over M called the *lifting bundle gerbe* of P. It is shown in [11] that the lifting bundle gerbe is trivial if and only if the bundle P lifts to a $\hat{G}$ bundle. We shall see below that every gerbe is stably isomorphic to a lifting bundle gerbe.

Note that the Chern class of $\hat{G} \to G$ defines a class in $H^2(G, \mathbb{Z})$. If G is connected then this induces a unique class in $H^2(P_m, \mathbb{Z})$ where $P_m = \pi^{-1}(m)$ for any m. The transgression map [14] maps this to a class in $H^3(M, \mathbb{Z})$. We have from [4]

Proposition 6.1. *The transgression of the class in $H^2(P_m, \mathbb{Z})$ defined by the Chern class of the bundle $\hat{G} \to G$ is the negative of the Dixmier-Douady class of the bundle gerbe $(\tilde{P}, P)$.*

6.2 $B\mathbb{C}^\times$ bundles

In [7] Gajer considers the groups $B^p\mathbb{C}^\times$ which are the p-fold classifying spaces of $\mathbb{C}^\times$. To do this it is necessary to use Milgram's construction of classifying spaces which has the property that if G is an abelian group then BG is an abelian group. It follows that all the $B^p\mathbb{C}^\times$ are (abelian) groups. It is also true that each $EB^{p-1}\mathbb{C}^\times$ is a group and there is a short exact sequence

$$0 \to B^{p-1}\mathbb{C}^\times \to EB^{p-1}\mathbb{C}^\times \to B^p\mathbb{C}^\times \to 0$$

of groups.

Gajer shows that the isomorphism classes of $B^p\mathbb{C}^\times$ bundles over a space M are in bijective correspondence with the cohomology $H^{p+2}(M, \mathbb{Z})$. In particular $H^3(M, \mathbb{Z})$ is in bijective correspondence with the space of $B\mathbb{C}^\times$ bundles on M. We will show

Theorem 6.2. *The set of all stable isomorphism classes of bundle gerbes is in bijective correspondence with the set of all isomorphism classes of $B\mathbb{C}^\times$ bundles on M*

Proof. Because we have the short exact sequence

$$0 \to \mathbb{C}^\times \to E\mathbb{C}^\times \to B\mathbb{C}^\times \to 0 \tag{5}$$

we can associate to any $B\mathbb{C}^\times$ bundle X over M the equivalence class of its lifting bundle gerbe. We want to show that this map is a bijection. It is enough to show that the characteristic class of a $B\mathbb{C}^\times$ bundle and the Dixmier-Douady class of its associated lifting bundle gerbe are the same.

The characteristic class of a $B\mathbb{C}^\times$ bundle is the pull-back of the generator of $BB\mathbb{C}^\times$ by any classifying map. Because $B^p\mathbb{C}^\times$ is a $K(\mathbb{Z}, p+1)$ the transgression for the fibering $EB\mathbb{C}^\times \to BB\mathbb{C}^\times$ is an isomorphism $H^2(B\mathbb{C}^\times, \mathbb{Z}) \to H^3(BB\mathbb{C}^\times, \mathbb{Z})$. Fix a generator of $H^3(BB\mathbb{C}^\times, \mathbb{Z})$ by defining it to be the negative of the transgression of the Chern class of $E\mathbb{C}^\times \to B\mathbb{C}^\times$. A choice of classifying map defines a commutative diagram

$$\begin{array}{ccc} X & \to & EB\mathbb{C}^\times \\ \downarrow & & \downarrow \\ M & \to & BB\mathbb{C}^\times. \end{array}$$

It follows that transgression maps of these two fiberings will commute with pull-backs and hence that the characteristic class of the bundle $X \to M$ is the negative of the transgression of the generator of two-dimensional cohomology in any of the fibres. The required result follows from 6.1. ∎

7 Classifying theory for bundle gerbes

The classifying theory for bundle gerbes now follows from that for $B\mathbb{C}^\times$ bundles. We define the universal bundle gerbe to be the lifting bundle gerbe for the fibration $EB\mathbb{C}^\times \to BB\mathbb{C}^\times$ and we have

Theorem 7.1. *If (P, Y) is a bundle gerbe over M there is a map $f\colon M \to BB\mathbb{C}^\times$, unique up to homotopy, such that the pull-back of the universal bundle gerbe is stably isomorphic to (P, Y).*

Note that this theorem has as corollaries a number of results about the stable isomorphism classes of bundle gerbes. For example it shows that every bundle gerbe is stably isomorphic to a lifting bundle gerbe and that every bundle gerbe is stably isomorphic to a bundle gerbe (P, Y) where the map $Y \to M$ is a fibration with fibre $K(\mathbb{Z}, 2)$.

Given a class in $H^3(M, \mathbb{Z})$ we can (following Gajer) use the exactness of (5) to construct isomorphisms of sheaf cohomology groups:

$$H^3(M, \mathbb{Z}) \simeq H^2(M, \underline{\mathbb{C}^\times}) \simeq H^1(M, \underline{B\mathbb{C}^\times}) \simeq H^0(M, \underline{BB\mathbb{C}^\times})$$

where the last of these spaces is the space of all maps from M to $BB\mathbb{C}^\times$. Here we use the notation $\underline{G}$ for the sheaf of maps into the group G. This gives a correspondence between the Dixmier-Douady class of a bundle gerbe and the classifying map of the associated $B\mathbb{C}^\times$ bundle. Explicit formulae for these maps, at the level of cocycles, are given in [7] Remark 2.1.

7.1 Lifting bundle gerbes

As an example consider the case of a lifting bundle gerbe. There we start with a principal G bundle P over M where G has a central extension:

$$0 \to \mathbb{C}^\times \to \hat{G} \to G \to 0.$$

To this we have seen there is associated a bundle gerbe and hence a $B\mathbb{C}^\times$ bundle. On the other hand the classifying map for the $\mathbb{C}^\times$ bundle $\hat{G} \to G$ is a map

$$\chi\colon G \to B\mathbb{C}^\times.$$

If χ was a group homomorphism then standard techniques would enable us to define an induced $B\mathbb{C}^\times$ bundle as $\mathcal{P} = P \times_G B\mathbb{C}^\times$. Such a χ exists if we work with geometric realisations of BG instead of the fat realisations. This has the advantage [13] that if G is any group then EG is a group and $G \to EG$ is a homomorphism. In particular $\hat{G} \to E\hat{G}$ is a homomorphism and hence the induced map $G \to E\hat{G}/\mathbb{C}^\times$ is a homomorphism. But because $E\hat{G}$ is contractible we must have $E\hat{G}/\mathbb{C}^\times$ homotopy equivalent to $B\mathbb{C}^\times$. For the purposes of the remaining discussion we take $E\hat{G}/\mathbb{C}^\times$ to be the realisation of $B\mathbb{C}^\times$. We conclude that $B\mathbb{C}^\times$ is a group and that we have defined a homomorphism $\chi\colon G \to B\mathbb{C}^\times$. Because of the commuting diagram

$$\begin{array}{ccc} \hat{G} & \to & E\hat{G} \\ \downarrow & & \downarrow \\ G & \stackrel{\chi}{\to} & B\mathbb{C}^\times \end{array}$$

we see that the homomorphism $G \to B\mathbb{C}^\times$ is a classifying map for the bundle $\hat{G} \to G$.

The homomorphism $G \to B\mathbb{C}^\times$ can be used to define an associated $\mathbb{C}^\times$ bundle to P by defining $\mathcal{P} = P \times_G B\mathbb{C}^\times$. If we choose a classifying map $\phi\colon M \to BG$ for the bundle $P(M, G)$ then we have a commuting diagram

$$\begin{array}{ccccc} P & \to & EG & \to & EB\mathbb{C}^\times \\ \downarrow & & \downarrow & & \downarrow \\ M & \stackrel{\phi}{\to} & BG & \stackrel{B\chi}{\to} & BB\mathbb{C}^\times \end{array}$$

The composition $B\chi \circ \phi$ is a classifying map for the associated bundle $\mathcal{P}$ which is the $B\mathbb{C}^\times$ bundle associated to the lifting bundle gerbe of P.

Note finally that if $G = LK$ is the group of smooth loops into a compact Lie group then it is well-known that there is a commuting diagram of $U(1)$ bundles

$$\begin{array}{ccc} \hat{G} & \rightarrow & U(\mathcal{F}) \\ \downarrow & & \downarrow \\ G & \rightarrow & PU\mathcal{F} \end{array}$$

where $\hat{G}$ is the Kac-Moody central extension, $U(\mathcal{F})$ the unitary operators on the Fock space $\mathcal{F}$, $PU(\mathcal{F})$ the projectivization of this group and the horizontal maps are all group homomorphisms [12]. Because $U(\mathcal{F})$ is contractible we see that $PU(\mathcal{F})$ is a realisation of $PU(1)$. This gives a concrete realisation of the $BU(1)$ bundle associated to an LK bundle.

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